Geometry of foliations of Minkowski spaces

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Abstract. As is well known, foliations of constant curvature and foliations generated by the orbits of Killing vector fields are important classes of foliations from a geometric point of view. The paper studies the geometry of some foliations of the Minkowski space, which arise in a natural way. It is shown that these foliations are foliations of constant curvature. The paper also studies the geometry of some singular foliations generated by the orbits of Killing vector fields.

1 Introduction

The Minkowski space can serve as a model of the space-time of the special theory of relativity. The Minkowski space is the basic space model of quantum physics that plays an important role in general relativity. In recent years, with the development of the theory of relativity, physicians and geometers extended the topics in classical ifferential geometry of Riemannian manifolds to that of Lorentzian manifolds. It is clearly demonstrated by the fact that many works in Euclidean space have found their counterparts in Minkowski space [1-6]. A. Ya. Narmanov and Zh. Aslonov obtained a complete classification of the singular Riemannian foliations of three-dimensional Euclidean space generated by the orbits of Killing vector fields[9]. The reachability set geometries of the Killing family of vector fields were studied by S.S.Saitova [10]. Minkowski space and its sub-space geometries are well studied. Its differential geometry [1-4,13-15] is given in these works
1 Basic concepts of pseudo-Euclidean spaces and foliation theory.
Let V_3 denote the real vector space with its usual vector structure. Denote by $B=\{e_1,e_2,e_3\}$ the canonical basis of R_3, that is

$$e_1=(1,0,0),\ \ e_2=(0,1,0),\ \ e_3=(0,0,1).$$

We denote (x, y, z) the coordinates of a vector with respect to B. We also consider in R_3 its affine structure, and we will say “horizontal” or “vertical” in its usual sense.
We say the scalar product $(V_3,\langle,\rangle)$ of the vectors $X\{x_1,y_1,z_1\}$ and $Y\{x_2,y_2,z_2\}$ the number defined by the following rule:

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$$\langle X,Y\rangle = x_1x_2 + y_1y_2 - z_1z_2 \tag{1}$$

and the exterior product by

$$[X,Y] = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} e_1 + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} e_1 - \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} e_1 \tag{2}$$

Definition 1. Vector space V_3, in which the scalar product of vectors is defined by formula (1), is said to be the Minkowski space ${}^1R_3$ [3].

We also use the terminology Minkowski space and Minkowski metric to refer the space and the metric, respectively. The Minkowski metric is a non-degenerate metric of index 1.

Definition 2. A vector $v \in {}^1R_3$ is said

(1) spacelike if $\langle v,v\rangle > 0$ or $v = 0$,

(2) timelike if $\langle v,v\rangle < 0$ and

(3) lightlike if $\langle v,v\rangle = 0$ and $v = 0$.[14]

The norm of a vector $|\vec{x}|$ is defined as the square root of the scalar square of the vector, and the distance between two points is defined as the norm of the vector connecting these points.

In Minkowski space ${}^1R_3$, two surfaces play the same role as spheres in R_3: the pseudohyperbolic surface and the pseudosphere. The pseudohyperbolic surface of radius $r > 0$ is the quadric

$$H(r) = \{p \in {}^1R_3;\ \langle p,\ p\rangle\ = -r^2\}.$$

This surface is spacelike. From the Euclidean viewpoint, $H(r)$ is the hyperboloid of two sheets $x_1^2 + x_2^2 - x_3^2 = -r^2$ which is obtained by rotating the hyperbola $x_1^2 - x_3^2 = -r^2$ in the plane $x_2 = 0$ with respect to the $x_3 -$ axis. The second surface is the pseudosphere or Lorentz sphere $S(r)$:

$$S(r) = \{p \in {}^1R_3;\ \langle p,\ p\rangle = r^2\}.$$

This surface is time-like and obtained by rotating the hyperbola $x_1^2 - x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the $x_3 -$ axis.

The lightlike cone of center p_0 is

$$C(r) = \{p \in {}^1R_3;\ \langle p - p_0,\ p - p_0\rangle = 0\} \setminus \{p_0\}.$$

Let M be a nondegenerate connected surface in ${}^1R_3$. The Gauss curvature of M is defined of a local parametrization $X(u,v)$, K is given by

$$K = \frac{NL - M^2}{EG - F^2}.$$

Where E, F, G and N, M, L are the coefficients of the first and second fundamental forms.
Recall that $EG-F^2>0$ if M is spacelike and $EG-F^2<0$ if M is timelike.

We present the definition of a foliation given in [8].
Let M be a smooth connected manifold of dimension n. Smoothness in this paper means the class C^∞ -smoothness.
Let us recall the definition of a foliation.

Definition 3. A foliation $F=\{L_\alpha;\alpha\in B\}$ on M of dimension k (codimension $(n-k)$) is a partition of M into arcwise connected subsets L_α with the following properties:
1. $M=\bigcup L_\alpha$,
2. for all $\alpha,\beta\in B$ if $\alpha\neq\beta$, then $L_\alpha\bigcap L_\beta=\varnothing$
3. For every point $p\in M$ there is an open neighborhood U of p and a chart $x=(x_1,x_2,\dots,x_k,y_1,y_2,\dots,y_{(n-k)})$ such that for each leaf L_α the connected components of $L_\alpha\cap U$ are defined by the equations $y_1=const$, $y_2=const$, $\cdots$, $y_{n-k}=const$. Such a chart is a distinguished chart.

The connected components of the sets $y_1=const, y_2=const,\dots,y_{n-k}=const$ in a distinguished chart are called plaques (plates) of F. Fixing $y_1=const, y_2=const,\dots,y_{n-k}=const$, the map $x\to(x,y))$ is a smooth embedding, therefore the plaques are connected $k-$dimensional submanifolds of M. This shows that each leaf L_α is a union of plaques and there exists a differential structure σ_α on L_α such that (L_α,σ_α) is a $k-$dimensional connected manifold. Note that the canonical injection $i\ :(L_\alpha,\sigma_\alpha)\to M$ is an immersion, but it is not necessarily an embedding [16].

Example 1. The simplest example of 2 - dimensional foliation is the representation of the Minkowski space ${}^1R_3$ as unions of 2 - dimensional parallel planes along the x_1-axis..

Definition 4. A partition F of a manifold M into leaves is called a smooth (from the class C^r) singular foliation (that is, a foliation with singularities) if the following conditions are satisfied:
1. for each point $x\in M$ there is a C^r map (U,φ) containing point x such that $\varphi(U)=V_1\times V_2$ where $V_1-$is the origin neighborhood in R^k, V_2- is the origin neighborhood in R^{n-k}, $k-$ is the dimension of the layer passing through the point x ;
2. $\varphi(x)=(0,0)$;
3. for each layer L_α such that $L_\alpha\cap U\neq\varnothing$, equality $L_\alpha\cap U=\varphi^{-1}(V_1\times l)$ holds, where $l=\{v\in V_2:\varphi^{-1}(0,v)\in L_\alpha\}$.

If the dimensions of the leaves of a foliation with singularities are the same, then it is a regular foliation in the sense of the definition given in [].

Let X - a vector field on M, $x \in M$, $X^t(x)$ be an integral curve of the vector field X passing through the point x at $t = 0$. The mapping $t \to X^t(x)$ is defined in some region $I(x)$, which generally depends not only on the field X, but also on the starting point x. In what follows, everywhere in formulas of the form $X^t(x)$ we will assume that $t \in I(x)$. If for all points $x \in M$ the domain $I(x)$ of the curve $t \to X^t(x)$ coincides with the real axis, then the vector field X is called a complete vector field.

Definition 5. The vector field X on M is called a Killing vector field if the one-parameter group of local transformations $t \to X^t(x)$ generated by the field X consists of motions (isometries) of the manifold M.

Definition 6. The orbit $L(x)$ of the family D of vector fields passing through the point x is defined (see [10]) as the set of such points y out of M for which there are real numbers $t_1, t_2, \ldots, t_k$ and vector fields $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ out of D (where k is an arbitrary natural number) such that

$$y = X_{i_k}^{t_k}\left(X_{i_{k-1}}^{t_{k-1}}\left(\ldots\left(X_{i_1}^{t_1}(x)\right)\ldots\right)\right).$$

2 Main results

Surfaces of constant curvature in Minkowski space We consider the spacelike ($x_3 > 0$) subspace of Minkowski as a manifold M.

Theorem 1. A two-dimensional foliation is a representation of a spacelike (timelike) space in the form of concentric pseudosphere (pseudohyperbolic) surfaces.

Proof.

Let's look at $x_3 > 0$ as manifolds M.

We will consider the $F = \{L_\alpha; \alpha \in B\}$ family, its element $L_\alpha = \{H(O):\ r = Const < 0\}$.

The pseudosphere at the beginning of the $H(O)$ center coordinate. L_α satisfies all the conditions of definition 2. This will be $F = \{L_\alpha; \alpha \in B\}$ a two-dimensional foliation.

The same is true for timelike space. The proof of Theorem 1 is complete.

Let $D = \{X\ , Y\}$ be a family of smooth vector fields in ${}^1R_3$, where

$$X = \{-x_2; x_1; 0\},\quad Y = \{0, x_3, x_2\}.$$

The vector field $X = \{-x_2; x_1; 0\}$ generates the following one-parameter transformation group

$$X^t : (x_1, x_2, x_3) \to \{x_1 \cos t - x_2 \sin t; x_1 \sin t + x_2 \cos t; x_3\},\ \ t \in R.$$

The flow of the vector field $Y = \{0, x_3, x_2\}$ consists of the following transformations

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$$Y^s:(x_1,x_2,x_3)\to\{x_1;x_2\cosh s+x_3\sinh s;x_2\sinh s+x_3\cosh s\}\quad s\in R.$$

We have the following theorem, which explains the geometry of this foliation.

Theorem 2. The orbits of the family of vector fields $D=\{X\ ,Y\}$ generate a singular foliation whose singular leaf is a point, and whose regular leaves are a cone $C(O)$ with a vertex punctured at the origin, Lorentz sphere $S(r)$ and pseudohyperbolic surface $H(r)$.

Proof. The origin of coordinates is a fixed point of the flows of both vector fields.

Thus, one of the orbits is the origin, i.e. $L_1=\{(0;0;0)\}$.

Function $F(x_1,x_2,x_3)=x_1^2+x_2^2-x_3^2$ is an invariant function for groups of transformations generated by vector fields and X, Y since $X(F)=0,\ Y(F)=0$. Therefore, the level surfaces of this function are invariant sets for transformations generated by the vector fields X, Y.

The level surface

$$L_2^+=\{(x_1;x_2;x_3):x_1^2+x_2^2-x_3^2=0,\ x_3>0\},$$

is the upper part of the cone with the apex punched out.

We take two arbitrary points from the set L_2^+, i.e. $A_1(x_{1,1};x_{2,1};x_{3,1}),A_2(x_{1,2};x_{2,2};x_{3,2})\in L_2^+$. Let us show that there are parameters t, s such that the following equality

$$Y^s(X^t(A_1))=A_2$$

holds.

We rewrite this relation in the coordinate form:

$$\begin{cases} x_{1,1}\cos t-x_{2,1}\sin t=x_{1,2}\\ (x_{1,1}\sin t+x_{2,1}\cos t)\cosh s+x_{3,1}\sinh s=x_{2,2}\\ (x_{1,1}\sin t+x_{2,1}\cos t)\sinh s+x_{3,1}\cosh s=x_{3,2}\end{cases}$$

From here we find

$$t=\arccos\left(\frac{x_{1,2}}{\sqrt{x_{1,2}^2+x_{2,2}^2}}\right)-\arccos\left(\frac{x_{1,1}}{\sqrt{x_{1,2}^2+x_{2,2}^2}}\right)+2\pi k,\quad k\in N$$

$$s=\ln\sqrt{\frac{(x_{2,2}+x_{3,2})(x_{3,1}-(x_{1,1}\sin t+x_{2,1}\cos t))}{(x_{2,2}-x_{3,2})(x_{3,1}+(x_{1,1}\sin t+x_{2,1}\cos t))}}.$$

Hence, there are values of the parameters t,s, i.e. from any point $A_1(x_{1,1};x_{2,1};x_{3,1})\in L_2^+$ with the help of flows of vector fields $X=\{-x_2;x_1;0\}$, $Y=\{0,x_3,x_2\}$ it is possible to move to any other point $A_2(x_{1,2};x_{2,2};x_{3,2})\in L_2^+$, therefore the set L_2^+ is an orbit of the family of vector fields D.

It is proved similarly that the set $L_2^- = \{(x_1; x_2; x_3) : x_1^2 + x_2^2 - x_3^2 = 0,\ x_3 > 0\}$, is also an orbit of the family of vector fields D .

Now consider the level surface

$$L_3 = \{(x_1; x_2; x_3) : x_1^2 + x_2^2 - x_3^2 = c,\ \ c > 0\},$$

of the invariant function $F(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$.

Let us show that from the point $A_1(x_{1,1}; x_{2,1}; 0)$ of the surface L_3 one can get to any point $A_2(x_{1,2}; x_{2,2}; x_{3,2})$ of the surface L_3 .

If the point $A_1(x_{1,1}; x_{2,1}; 0)$ moves along the integral curve of the vector field $Y = \{0, x_3, x_2\}$, then in time s it arrives at the point with coordinates $(x_1(s); x_2(s); x_3(s))$, where

$$\begin{cases} x_1(s) = x_{1,1} \\ x_2(s) = x_{2,1}\cosh s \\ x_3(s) = x_{2,1}\sinh s \end{cases}.$$

Now let's show that there is a value of s for which $x_3(s) = x_{3,2}$.

The last equality is equivalent to $x_{2.1}\sinh s = x_{3,2}$,stems from:

$$s = \arcsin h\left(\frac{x_{3,2}}{x_{2,1}}\right).$$

There is a value of s for which $x_3(s) = x_{3,2}$.

The point with coordinates $(x_{1,1}; x_2(s); x_{3,2}\}$ along the integral curve of the vector field $X = \{-x_2; x_1; 0\}$ can be transferred to the point with coordinates $A_2(x_{1,2}; x_{2,2}; x_{3,2})$.

Thus the level surface L_3 , is an orbit of the family of vector fields D . This surface is Lorentz sphere $S(r)$.

We will show that the upper part of the

$$L_4^+ = \{(x_1; x_2; x_3) : x_1^2 + x_2^2 - x_3^2 = c,\ \ x_3 > c\},$$

pseudohyperbolic surface $H(r)$ is an orbit. Let $A_1(0; 0; x_{3,1})$ be the vertex of the upper part of the two-sheeted hyperboloid. Here $x_{3,1} \geq -c$. Let us show that from the point $A_1(0; 0; x_{3,1})$ of the surface L_4^+ one can get to any other point $A_2(x_{1,2}; x_{2,2}; x_{3,2})$ of the surface L_4^+.

If the point $A_1(0;0;x_{3,1})$ moves along the integral curve of the vector field $Y=\{0,x_3,x_2\}$, then in time s it arrives at the point with coordinates $(x_1(s);x_2(s);x_3(s))$, where

$$\begin{cases} x_1(s)=0 \\ x_2(s)=x_{3,1}\sinh s \\ x_3(s)=x_{3,1}\cosh s \end{cases}.$$

Let us show that there exists a value of s for which $x_3(s)=x_{3,1}$.
The equality $x_3(s)=x_{3,1}$ is equivalent to the quadratic equation $x_{3,1}\cosh s=x_{3,2}$,

$$s=\operatorname{arccos}h\left(\frac{x_{3,2}}{x_{3,1}}\right).$$

There is a value of s for which $x_3(s)=x_{3,2}$.

The point with coordinates $(x_{1,1};x_2(s);x_{3,2}\}$ along the integral curve of the vector field $X=\{-x_2;x_1;0\}$ can be transferred to the point with coordinates $A_2(x_{1,2};x_{2,2};x_{3,2})$.
It is proved similarly that the lower part of pseudohyperbolic surface $H(r)$

$$L_4^-=\left\{(x_1;x_2;x_3):x_1^2+x_2^2-x_3^2=c,\ \ c<0\ \ x_3<c\right\},$$

is also an orbit. The proof of Theorem 2 is complete.

Consider the four-dimensional Minkowski space. The following 10 vector fields will be the Killing vector field.

$$X_1=\{-x_2,x_1,0,0\},X_2=\{0,-x_3,x_2,0\},X_3=\{-x_3,0,x_1,0\},$$
$$X_4=\{0,x_4,0,x_2\},X_5=\{0,0,x_4,x_3\},X_6=\{x_4,0,0,x_1\},$$
$$X_i=\frac{\partial}{\partial x_i},i=7,8,9,10$$

Each vector fields $X_i,i=1,2,...,10,$ respectively generates the following one-parameter group of transformations.
Euclidean rotation:

$$X_1^t:(x_1;x_2;x_3;x_4)\to\{x_1\cos t-x_2\sin t;x_1\sin t+x_2\cos t;x_3;x_4\},$$
$$X_2^t:(x_1;x_2;x_3;x_4)\to\{x_1;x_2\cos t-x_3\sin t;x_2\sin t+x_3\cos t;x_4\},$$
$$X_3^t:(x_1;x_2;x_3;x_4)\to\{x_1\cos t-x_3\sin t;x_2;x_1\sin t+x_3\cos t;x_4\},\ \ t\in R.$$

Pseudo-rotations:

$$X_4^t:(x_1,x_2,x_3,x_4)\to\{x_1;x_2\cosh t+x_4\sinh t;x_3;x_2\sinh t+x_4\cosh t\},$$
$$X_5^t:(x_1,x_2,x_3,x_4)\to\{x_1;x_2;x_3\cosh t+x_4\sinh t;x_3\sinh t+x_4\cosh t\},$$
$$X_6^t:(x_1,x_2,x_3,x_4)\to\{x_1\cosh t+x_4\sinh t;x_2;x_3;x_1\sinh t+x_4\cosh t\},\ t\in R.$$

Parallel transmission:

$$X_7^t:(x_1;x_2;x_3;x_4)\to\{x_1+t;x_2;x_3;x_4\},X_8^t:(x_1;x_2;x_3;x_4)\to\{x_1;x_2+t;x_3;x_4\},$$
$$X_9^t:(x_1;x_2;x_3;x_4)\to\{x_1;x_2;x_3+t;x_4\},X_{10}^t:(x_1;x_2;x_3;x_4)\to\{x_1;x_2;x_3;x_4+t\},\ t\in R.$$

Consider the orbit of a family of vector fields in four-dimensional Minkowski space ${}^1R_4$. The orbits of the family of Killing vector fields $D_1=\{X_1,X_2,X_4\}$ generate a singular foliation whose singular leaf is a point, and whose regular leaves are a cone $C(O)=\{P\in{}^1R_4\colon\langle\overrightarrow{OP},\ \overrightarrow{OP}\rangle=0\}\setminus\{O\}$ with a vertex punctured at the origin, Lorentz sphere $S(r)=\{P\in{}^1R_4\colon\langle\overrightarrow{OP},\ \overrightarrow{OP}\rangle=r^2\}$ and pseudohyperbolic surface $H(r)=\{P\in{}^1R_4\colon\langle\overrightarrow{OP},\ \overrightarrow{OP}\rangle=r^2\}$, here O is the origin of the coordinate system.. Note that the following triples of vector fields also generate the considered foliation:

$$D_2=\{X_1,X_3,X_4\},D_3=\{X_2,X_3,X_4\},D_4=\{X_1,X_3,X_5\},$$
$$D_5=\{X_2,X_3,X_5\},D_6=\{X_1,X_3,X_6\},D_7=\{X_2,X_3,X_6\}.$$

3 Conclusion

The paper deals with foliations of constant curvature of the Minkowski space, which are analogues of the family of concentric spheres in the Euclidean space. The geometry of the singular foliation generated by the orbits of Killing vector fields is also studied.

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